ON THE ISOMORPHISM PROBLEM FOR MEASURES ON BOOLEAN ALGEBRAS

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Abstract. The paper investigates possible generalisations of Maharam's theorem to a classification of Boolean algebras that support a finitely additive measure. We prove that Boolean algebras that support a finitely additive non-atomic uniformly regular measure are metrically isomorphic to a subalgebras of the Jordan algebra with the Lebesgue measure. We give some partial analogues to be used for a classification of algebras that support a finitely additive non-atomic measure with a higher uniform regularity number. We show that some naturally induced equivalence relations connected to metric isomorphism are quite complex even on the Cantor algebra and therefore probably we cannot hope for a nice general classification theorem for finitely–additive measures.

We present an example of a Boolean algebra which supports only separable measures but no uniformly regular one.

1. Introduction

A celebrated result in measure theory is the theorem of Maharam [20] which states that if μ is a homogeneous σ-additive measure on a σ-complete Boolean algebra $\mathfrak{B}$, then the measure algebra of $(\mathfrak{B},\mu)$ is isomorphic to the measure algebra of some 2^κ with the natural product measure. Moreover, every measure algebra can be decomposed into a countable sum of such algebras where the measure is homogeneous. This provides us with a very beautiful classification of σ-complete measure algebras, hence it is natural to ask if a similar characterisation can be obtained under weaker assumptions. In particular, a natural class to consider is formed by pairs $(\mathfrak{B},\mu)$ where $\mathfrak{B}$ is any Boolean algebra, not necessarily σ-complete, and μ is a *strictly positive finitely additive* measure on $\mathfrak{B}$, where μ without loss of generality assigns measure 1 to the unit element of $\mathfrak{B}$. In her commentaries to the 1981 edition of the "Scottish book" D. Maharam posed this question.

A closely connected, but not the same problem, is that of obtaining a combinatorial characterisation of Boolean algebras $\mathfrak{B}$ which *support a measure*, that is, for which there is a finitely additive μ which is strictly positive on $\mathfrak{B}$. It is easy to see that such an algebra must satisfy the countable chain condition ccc and the question of the sufficiency of this condition was raised by Tarski in [27]. Horn and Tarski in [13] suggested various other chain conditions and Gaifman in [9] showed that in fact a rather strong condition of being a union $\mathfrak{B}=\bigcup_{n<\omega}\mathcal{F}_n$ where each $\mathcal{F}_{n+1}$ is

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n-linked, does not suffice for $\mathfrak{B}$ to support a measure. Kelley in [16] gave an exact combinatorial characterisation of Boolean algebras that support a measure, which we therefore call Kelley algebras. This characterisation is that $\mathfrak{B} \setminus \{0\} = \bigcup_{n<\omega} \mathcal{F}_n$, where each $\mathcal{F}_n$ has a positive *Kelley intersection number*. The Kelley intersection number for a family $\mathcal{F}$ of sets is said to be $\geq \alpha$ if for every $n < \omega$ and every sequence of n elements of $\mathcal{F}$ (possibly with repetitions), there is a subsequence of length at least $\alpha \cdot n$ which has a nonempty intersection. Then the Kelley intersection number is the sup of all α such that the intersection number of is $\geq \alpha$. This characterisation is unfortunately not very useful in practice, as it is hard to check, but nevertheless, it sheds light on our initial problem of classification. Namely, it shows that every σ-centred Boolean algebra does support a measure.[1] The σ-centred Boolean algebras are exactly the subalgebras of $\mathcal{P}(\omega)$, which for various good reasons are considered to be unclassifiable.

This detour shows that if we hope to have a classification of Kelley algebras, we should better first restrict to some reasonable subclass. Maharam's theorem suggests that there should be some cardinal invariant at least as a first dividing line, so the natural first reduction is to consider only those Boolean algebras that support a *separable* measure. This can be easily defined by noticing that a strictly positive measure μ on a Boolean algebra $\mathfrak{B}$ induces a metric d given by $d(a,b) = \mu(a \triangle b)$. This gives rise to the cardinal characteristics given by the density character of this metric space, which is exactly the Maharam type of a measure algebra if the algebra is σ-closed and the measure σ-additive and homogeneous. The measure μ is said to be *separable* if the density character defined above is equal to ω. The question of a combinatorial characterisation of Boolean algebras that support a separable strictly positive measure has already been considered by many authors, including Talagrand in [26], who proposed a plausible candidate characterisation and showed that even so it can only be true consistently. In [5] Džamonja and Plebanek have shown that there is a ZFC counterexample to this characterisation, therefore putting the characterisation programme back to zero.

Going back to the fact that all subalgebras of $\mathcal{P}(\omega)$ are Kelley algebras, we note that it is rather easy to construct atomic separable measures: counting measures. Many subalgebras of $\mathcal{P}(\omega)$ only support such a measure, so it is more natural to restrict our attention to the non-atomic case. For a Boolean algebra to support a non-atomic measure it is of course necessary that the algebra itself be non-atomic, so we shall mostly consider such algebras. In [5] it is shown that Martin's axiom (for cardinals $< \mathfrak{c}$) implies that every non-atomic Boolean algebra of size less than $\mathfrak{c}$ supports a non-atomic separable measure, which shows on the one hand that the characterisation of algebras supporting a non-atomic separable measure is really about algebras of size $\mathfrak{c}$ (it is easy to see that such a measure cannot exists on an algebra of size $> \mathfrak{c}$), and on the other hand that a classification is difficult since it potentially includes all small enough non-atomic algebras.

Effros' paper [6] enunciated and Harrington–Kechris–Louveau paper [12] initiated a programme of a classification of equivalence relations on Polish spaces in order to use them as a measuring device for the classification difficulty of various problems naturally arising in mathematics. Namely, suppose that we wish to classify the objects in a certain class, say we are in a Polish space and we wish to classify definable subspaces of it according to some equivalence relation. The

[1] This can also be seen using the Stonre representation theorem.

equivalence relation may be understood as stating that any two equivalent objects have the same invariant. If this classification is useful, then the invariant should be definable and checking if two objects are in the same class should be doable in a definable way. If we show that such a definable classification is not possible, then we have shown that the class we started is unclassifiable in reasonable terms. Since the beginning of this programme there has emerged a powerful machinery which has been successfully used to show the complexity of various classification problems. In this paper our thesis is that the class of Boolean algebras supporting a separable measure is difficult to classify. To support it we would ideally like to use the descriptive set theoretic machinery mentioned above. However, we cannot approach this problem directly using the classification techniques of equivalence classes in Polish spaces, since the Boolean algebras in question cannot be coded as elements of a Polish space. On the other hand, since we are after a non-classification result, we may restrict to a subclass of our initial class, which may be seen as coming from a Polish space and show a non-classification of even that smaller class. This is exactly what we do, namely we simply consider the measures on the Cantor algebra, and we show that even in the class there the classification of measures up to isomorphism is basically Borel complete.

The above result shows that we cannot hope for a good classification theorem for finitely–additive measures. However, in the rest of the paper we argue that some partial classifications can be done. We cannot introduce a good invariant for classification but we can describe Boolean algebras with measures as subalgebras of some benchmark Boolean algebras satisfying some additional conditions. Restating the definition from the above, a measure μ on $\mathcal{B}$ is separable iff there is a countable $\mathfrak{A} \subseteq \mathfrak{B}$ which is *μ-dense* in the sense that for every $b \in \mathcal{B}$ and $\varepsilon > 0$ there is $a \in \mathfrak{A}$ such that $\mu(a \bigtriangleup b) < \varepsilon$. Let us say that $\mathfrak{A} \subseteq \mathfrak{B}$ is *μ-uniformly dense* if for every $b \in \mathcal{B}$ and $\varepsilon > 0$ there is $a \in \mathfrak{A}$ such that $a \leq b$ and $\mu(b \setminus a) < \varepsilon$. The *uniform density* of $(\mathfrak{B}, \mu)$ can be defined as the smallest cardinal κ such that there is a μ-uniformly dense $\mathfrak{A} \subseteq \mathfrak{B}$ of size κ. Measures whose uniform density is $\aleph_0$ are called *uniformly regular* and they have been considered in the literature, for example in [22], where it was shown that uniform regularity is quite different than separability. We show that in fact for the purposes of characterisation, uniformly regular measures are much superior to the separable ones, since our Theorem 4.9 gives that a Boolean algebra supports a non-atomic uniformly regular measure iff it is a subalgebra of the so called Jordan algebra, an algebra which is well known in the literature (see e.g. [14], [24]). The Jordan algebra can be defined as a maximal subalgebra of the Cohen algebra, on which the Lebesgue measure is σ–additive (on its domain). The Jordan algebra is a classical object that was introduced in the process of constructing the Jordan measure, in fact much before the Lebesgue measure. We prove that all non–atomic measures supported by the Jordan algebra are mutually isomorphic. Thus, we obtain something like Maharam theorem for uniformly regular measures: every uniformly regular non–atomic strictly positive measure defined on a "maximal possible" Boolean algebra is isomorphic to a Lebesgue measure on the Jordan algebra. [We recall in §4 that it is also known that algebras supporting a non-atomic separable measure are subalgebras of a fixed algebra (namely the Random one) but that there is no iff characterisation known.] Our results suggest that the classification of non-atomic Kelley algebras should proceed with the invariant being the uniform density rather than the Maharam type. Another conclusion is

that although measures on the Cantor algebra form a complicated structure with respect to isomorphism, they all have the same (up isomorphism) extension to a measure the Jordan algebra.

Moving on to a more general case, one should recall that in fact the separable case of Maharam's theorem was known earlier (it was used for example in classification theorems for commutative von Neumann algebras of operators on separable Hilbert spaces) so one can see that the countable=separable case is the first building block of the classification. We would like to use our results on the classification of uniformly regular measures as a building block for the full classification accordying to the cardinal invariant of uniform regularity. It would be our goal to have a full analogue of Maharam's theorem. For the moment we only have partial results about the higher-dimensional case. We present them in Section 5.

The above considerations naturally raise the question of the identity of the classes that we are considering: for example is it possible that in fact the Boolean algebras that support a uniformly regular measure are exactly those that support a separable one? It follows from Theorem 4.9 that the Cohen algebra does not support a non-atomic uniformly non-regular measure, while it easily follows from the Kelley criterion and the fact that the Cantor algebra is dense in the Cohen algebra that the Cohen algebra does support a separable measure. We go further and in Theorem 6.3 discover a Kelley algebra which only supports separable measures but not a uniformly regular one.

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2. Basic definitions

Throughout, κ stands for an infinite cardinal. By a *measure* we understand *a finitely additive* non-negative measure. If we say that a measure μ is σ–additive on $\mathfrak{B}$ we do not necessarily assume that $\mathfrak{B}$ is σ–complete. We will consider only totally finite measures, so without loss of generality we only work with probability measures. A measure on a Boolean algebra is *non–atomic* if for every $\varepsilon > 0$ there is a finite partition of unity into elements of measures at most ε.

We will denote elements of a Boolean algebra by small letters, the unity by $\mathbf{1}$ and zero by $\mathbf{0}$, unless we work with an algebra of sets.

The *Cantor algebra* $\mathfrak{A}_c$ is the only Boolean algebra, with respect to isomorphism, which is countable and atomless. By saying that a Boolean algebra *has a Cantor subalgebra* we understand that it has a subalgebra isomorphic to $\mathfrak{A}_c$. It will be convenient to see $\mathfrak{A}_c$ as the Boolean algebra of clopen subsets of 2^ω with the product topology. Define *the generating tree* $\mathcal{T}$ as

$$\mathcal{T} = \{[s] \colon s \in 2^{<\omega}\}$$

and notice that $\mathfrak{A}_c$ is generated by $\mathcal{T}$. We will often identify $\mathcal{T}$ with $2^{<\omega}$.

The *Cohen algebra* $\mathfrak{C}$ is the completion of $\mathfrak{A}_c$. In the above setting $\mathfrak{C}$ can be seen as the algebra Borel(2^ω)/Meager. *The Lebesgue measure* for us is the measure λ on $\mathfrak{A}_c$ defined on $\mathcal{T}$ by $\lambda([s]) = 1/2^n$ if $s \in 2^n$ and then extended to $\mathfrak{A}_c$ and further to Borel(2^ω). The *Random algebra* $\mathfrak{R}$ is the Boolean algebra Borel$(2^\omega)/\{B \subseteq 2^\omega \colon \lambda(B) = 0\}$.

We will say that a Boolean algebra *carries* a measure μ if μ is defined on the whole of $\mathfrak{A}$. If, additionally, μ is strictly positive on $\mathfrak{A}$, we will say that $\mathfrak{A}$ *supports* μ. By a *Kelley algebra* we mean a Boolean algebra supporting a measure.

By $\mathrm{Free}(\kappa)$ we denote the free algebra on κ generators. A Boolean algebra is *big* if it has a subalgebra isomorphic to $\mathrm{Free}(\omega_1)$. Boolean algebras which are not big are *small*. The Cantor algebra $\mathfrak{A}_c$ is isomorphic to $\mathrm{Free}(\omega)$.

We will say that a family $\mathcal{D} \subseteq \mathcal{A}$ is *μ–dense* in $\mathcal{A}$, if

$$\inf\{\mu(a \triangle d) \colon d \in \mathcal{D},\ a \in \mathcal{A}\} = 0.$$

A family $\mathcal{D} \subseteq \mathcal{A}$ is *uniformly μ–dense* in $\mathcal{A}$ if

$$\inf\{\mu(a \setminus d) \colon d \in \mathcal{D},\ a \in \mathcal{A},\ d \leq a\} = 0.$$

We will say that the measure μ on $\mathfrak{A}$ is *separable* if there is a countable μ–dense family $\mathcal{D} \subseteq \mathfrak{A}$ and the measure μ on $\mathfrak{A}$ is *uniformly regular* if there is a countable uniformly μ–dense family $\mathcal{D} \subseteq \mathfrak{A}$.

Given a Boolean algebra $\mathfrak{A}$ with a strictly positive measure μ, we can define the *Fréchet-Nikodym metric induced by μ* by:

$$d_\mu(a, b) = \mu(a \triangle b), \text{ for } a, b \in \mathfrak{A}.$$

Moreover, every function $f \colon \mathfrak{A} \to \mathfrak{B}$ which is a (metric) isomorphism is an isometry of $(\mathfrak{A}, d_\mu)$ and $(\mathfrak{B}, d_\nu)$. An isometry f between $(\mathfrak{A}, d_\mu)$ and $(\mathfrak{B}, d_\nu)$ is not necessarily a metric isomorphism ($f(\mathbf{0})$ is not necessarily equal to $\mathbf{0}$), but the function $g(a) = f(a) \triangle f(\mathbf{0})$ is:

Proposition 2.1. *Suppose $\mathfrak{A}$ supports a measure μ and $\mathfrak{B}$ supports ν. If f is an isometry of $(\mathfrak{A}, d_\mu)$ and $(\mathfrak{B}, d_\nu)$, then the function g defined by $g(a) = f(a) \triangle f(\mathbf{0})$ is a metric isomorphism of $(\mathfrak{A}, \mu)$ and $(\mathfrak{B}, \nu)$.*

Proof. Using the fact that $\nu(g(a)) = \mu(a)$ for each $a \in \mathfrak{A}$ one can show that g is monotonic and if $a \wedge b = \mathbf{0}$, then $g(a) \wedge g(b) = \mathbf{0}$. The latter implies that $g(a^c) = g(a)^c$ and $\nu(g(a \vee b)) = \nu(g(a) \vee g(b))$ for each $a, b \in \mathfrak{A}$ such that $a \wedge b = \mathbf{0}$. From the monotonicity and the latter it follows that $g(a \vee b) = g(a) \vee g(b)$ for each $a, b \in \mathfrak{A}$. Hence, g is a metric isomorphism. □

Notice that a measure μ supported by $\mathfrak{A}$ is separable if and only if the space $(\mathfrak{A}, d_\mu)$ is separable.

We will consider equivalence relations on Polish spaces. We say that an equivalence relation E on a Polish space X is *reducible* to an equivalence relation F on a Polish space Y if there is a Borel function $f \colon X \to Y$ such that

$$x \mathsf{E} y \quad \text{iff} \quad f(x) \mathsf{F} f(y)$$

for each $x, y \in X$. Loosely speaking E is reducible to F if it is not more complex than F. There are several benchmark equivalence relations, which allow to place certain equivalence relation in the complexity hierarchy (see e.g. [10] or [15]). Particularly important is the notion of a Borel–complete (analytic–complete) equivalence relation, i.e. such that every Borel (analytic) equivalence relation can be reduced to it. An example of the former one is the isomorphism of countable graphs.

A Borel equivalence relation E is *smooth* if it is reducible to the relation of identity on a standard Borel space, the minimal equivalence relation with respect to reducibility among equivalence relations with uncountably many equivalence classes.

3. Equivalences of measures on the Cantor algebra

With the general aim of understanding the complexity of metric isomorphisms between Kelley algebras we here study the simplest possible (non–atomic) case: a metric isomorphism between measures defined on the Cantor algebra $\mathfrak{A}_c$.

We approach the question of the complexity of this relation by encoding measures in other mathematical objects. As we have noticed in Section 2, studying isomorphisms between measures on Boolean algebras is the same as investigating the isometry relation between (Fréchet–Nikodym) metric spaces. This is promising since the isometry of Polish spaces is a deeply explored equivalence relation. However we cannot use here directly its theory since it is unclear how to recognize Fréchet–Nikodym spaces within countable metric spaces and how the theory of complexity of isometries of Polish spaces can be used for countable metric spaces. We shall therefore use a different approach and study measures on the Cantor algebra. We will encode a strictly positive measure on Cantor algebra as a function from $2^{<\omega}$ to $(0,1)$. We note that studying measures on the Cantor algebra is not the same as studying measures on the Cantor space, although these two topics are quite related. Any measure on the Cantor algebra induces a measure on the Cantor space. Conversely, any measure on the Cantor space can be restricted to the algebra of clopen sets and is thus related to a measure on the Cantor algebra. Measures on the Cantor space and many related concepts are studied in S. Gao's book [10]. [2]

We consider three equivalence relations which are strictly weaker than the metric isomorphism. They are defined on three different subfamilies of measures on $\mathfrak{A}_c$ and they capture three different properties of measures: in Subsection 3.1 we deal with strictly positive measures and the equivalence relation induced by an automorphism of $2^{<\omega}$, in Subsection 3.2 we study all measures on $\mathfrak{A}_c$ and the relation induced by the isomorphism of ideals on $\mathfrak{A}_c$ and in Subsection 3.3 we investigate strictly positive non–atomic measures and the relation induced by the equality of ranges. These equivalence relations are in a way orthogonal to each other, i.e. there is no inclusion between any of their equivalence classes. None of the presented results can be used to show the complexity of the metric isomorphism in any of the above cases. Nevertheless, we try to draw some conclusions in Subsection 3.4.

3.1. **Strictly positive measures on $\mathfrak{A}_c$ and the relation induced by an automorphism on $2^{<\omega}$.** By [11] (cf. [3, Section 2]), given any isomorphic copy of the Cantor algebra, there is a Borel procedure of finding its generating tree. We can identify the copy of $\mathfrak{A}_c$ with $\mathfrak{A}_c$ itself and the generating tree with $2^{<\omega}$. In this setting, every measure on $\mathfrak{A}_c$ is uniquely determined by the values of this measure on $2^{<\omega}$. Hence, every measure supported by $\mathfrak{A}_c$ is uniquely defined by an element of $(0,1)^{2^{<\omega}}$. Of course, not every function defined on $2^{<\omega}$ can be extended to a measure on $\mathfrak{A}_c$ but in fact we can treat every element of $(0,1)^{2^{<\omega}}$ as a strictly positive measure on $\mathfrak{A}_c$ by using the following Borel coding. Given $f\colon 2^{<\omega} \to (0,1)$ and $s \in 2^{<\omega}$ let

$$\mu_f(s) = \prod_{s(n)=0} f(s|n) \cdot \prod_{s(n)=1} (1 - f(s|n)).$$

[2]Note that some of the terminology used in [] is different than the one employed here, in particular our notion of a strictly positive measure on the Cantor algebra is called 'non-atomic' in [10].

So, we will treat the space of strictly positive measures on $\mathfrak{A}_c$ as $\mathsf{spM} = (0,1)^{2^{<\omega}}$ with the standard topology. It is a standard Borel space. Every element of spM uniquely defines a strictly positive measure, but a strictly positive measure can be coded in many elements of spM.

We can then see the metric isomorphism as an equivalence relation on the standard Borel space. Namely, we can consider the equivalence relation $\equiv$ on spM induced by the metric isomorphism. Unfortunately, it is unclear to us how we can reduce $\equiv$ to an equivalence relation whose complexity is known, or which known equivalence relations can be reduced to $\equiv$, although it looks like $\equiv$ should be Borel complete. We can however say something about other equivalence relations connected to the metric isomorphism.

Denote by $\equiv_c$ the following equivalence relation on spM:

$$f \equiv_c g \text{ if there is an automorphism } \varphi \text{ of } 2^{<\omega} \text{ such that } \forall s \in 2^{<\omega} f(s) = g(\varphi(s)).$$

Notice that $f \equiv_c g$ implies $f \equiv g$ (but the reverse implication does not hold).

As $\equiv_c$ is defined by an automorphism of a compact Polish group, it follows from the known results in the theory of equivalence relations that it is a smooth Borel equivalence relation. We give a direct proof which also allows us to compare $\equiv_c$ with a well studied equivalence. Namely, denote by $\cong$ the isomorphism of countable graphs. Note that $\cong$ considered on the space of finitely branching trees is a Borel equivalence relation (cf. [8, Theorem 1.1.3]).

Theorem 3.1. *$\equiv_c$ is Borel reducible to the isomorphism of finitely branching trees.*

Proof. We describe how to code an element of $(0,1)^{2^{<\omega}}$ in a finitely branching tree (in a Borel way).

Every real $r \in (0,1)$ (in fact we will treat r as an element of 2^ω) can be coded in a finitely branching tree, e.g. in the following way. Start with a countable tree ω with the relations given by $n \mathsf{R} m$ if and only if $m = n+1$ and then add a terminal node to m-th vertex if $r(m) = 1$.

Fix a function $f\colon 2^{<\omega} \to (0,1)$. Start with $T_0 = 2^{<\omega}$ and then stick to every $s \in T_0$ the graph coding $f(s)$. We will denote the function described in this way by Ψ. Notice that Ψ is Borel.

Denote by $\cong$ the isomorphism relation on countable graphs. We will show that $\Psi(f) \cong \Psi(g)$ if and only if $f \equiv_c g$.

Assume that there is an isomorphism of graphs $F\colon \Psi(f) \to \Psi(g)$. No vertex s from T_0 can be mapped to a vertex from a code of a real, since s has infinitely many pairwise *disjoint* vertices below, contrary to the vertices in the codes of reals. Thus, $F[T_0] = T_0$ and $F|T_0$ has to be an automorphism. Moreover, two codes of reals are isomorphic only if they code the same real, so finally $f \equiv_c g$. On the other hand, an isomorphism $G\colon f \to g$ witnessing $f \equiv_c g$ clearly induces an isomorphism between $\Psi(f)$ and $\Psi(g)$.

Thus, $\equiv_c$ is Borel reducible to $\cong$ on finitely–branching trees. □

3.2. **All measures on $\mathfrak{A}_c$ and the relation induced by the isomorphism of ideals on $\mathfrak{A}_c$.** We can also code measures on $\mathfrak{A}_c$ which are not necessarily strictly positive. We can use the same formula as for strictly positive measures but we should consider only those functions from $2^{<\omega}$ to $[0,1]$ satisfying the (Π^0_2) condition:

$$\forall s \in 2^{<\omega} \ (f(s) = 0 \implies (f(s^\frown 0) = 0 \text{ and } f(s^\frown 1) = 0)).$$

Denote by M the space of elements of $[0,1]^{2^{<\omega}}$ satisfying the above condition. Notice that since M is a G_δ subspace of $[0,1]^{2^{<\omega}}$, it is a standard Borel space. We denote by $\equiv$ the equivalence relation on M induced by the metric isomorphism. Denote by $=^+$ the equivalence relation on $(0,1)^\omega$ of equality of countable subsets (i.e. $(x_n) =^+ (y_n)$ iff $\{x_n \colon n \in \omega\} = \{y_n \colon n \in \omega\}$). This is a Borel equivalence relation strictly simpler than graph isomorphism but still quite complex (see e.g. [10, Section 15.3]).

By $\equiv_m$ we mean the relation on M defined by

$$f \equiv_m g \text{ iff } (f(s) = 0 \iff g(s) = 0 \text{ for each } s \in 2^{<\omega}).$$

Clearly $f \equiv_m g$ if and only if the measures μ_f and μ_g are mutually absolutely continuous. This relation in the context of Cantor algebras is Π^0_2 and is actually smooth. In the context of Cantor spaces, it is a Π^0_3 relation and $=^+$ is reducible to $\equiv_m$ (see [10, Lemma 8.5.5]).

We are interested in a related equivalence relation:

$$f \equiv_z g \text{ if there is } g' \equiv g \text{ such that } f \equiv_m g'.$$

Theorem 3.2. *$\equiv_z$ is Borel complete.*

Proof. We will use the fact that the isomorphism relation of ideals on the Cantor algebra is Borel complete (see [3, Theorem 4]). Ideals on $\mathfrak{A}_c$ are isomorphic if there is an automorphism of $\mathfrak{A}_c$ sending one to another. Notice that ideals on $\mathfrak{A}_c$ are equal if they are equal on its generating tree.

Every proper ideal $\mathcal{I}$ on $\mathfrak{A}_c$ can be coded in a measure. We define the measure inductively with respect to the generating tree $2^{<\omega}$. For each $s \in 2^{<\omega}$

- if $\mu(s) = 0$, then $\mu(s^\frown 0) = \mu(s^\frown 1) = 0$;
- if $\mu(s) > 0$ and both $s^\frown 0$ and $s^\frown 1$ are not in $\mathcal{I}$, then let $\mu(s^\frown 0) = \mu(s^\frown 1) = 1/2\ \mu(s)$;
- finally, notice that if $\mu(s) > 0$ and for some $l < 2$, $s^\frown l \in \mathcal{I}$, then $s^\frown(1-l) \notin \mathcal{I}$. We define $\mu(s^\frown(1-l)) = \mu(s)$ and $\mu(s \wedge l) = 0$.

Now, encode the measure in M by the procedure described above. In this way we assign to each ideal $\mathcal{I}$ on $\mathfrak{A}_c$ an element $f_{\mathcal{I}}$ of M in such a (Borel) way that $f_{\mathcal{I}} \equiv_z f_{\mathcal{J}}$ if and only if the ideals are isomorphic. $\square$

3.3. **Strictly positive non–atomic measures on $\mathfrak{A}_c$ and relation induced by ranges of measures.** The basic property which allows to distinguish measures defined on $\mathfrak{A}_c$ is the range. If $\mathrm{rng}(\mu) \neq \mathrm{rng}(\nu)$, then μ and ν cannot be metrically isomorphic. So, it is tempting to code a measure in the countable subset of $[0,1]$ as its range. However, there are some reasons why this coding is not useful for our purposes. Firstly, only particular subsets of $[0,1]$ can serve as a range of a measure. Secondly, two non–isomorphic measures can have the same range.

Indeed, consider $\mathfrak{B} \subseteq \mathfrak{A}_c$ (treated here as $\mathrm{Clopen}(2^\omega)$) generated by $B_0 = [(0,0)]$, $B_1 = [(0,1)]$, $B_2 = [(1,0)]$ and $B_3 = [(1,1)]$. Let μ and ν be measures defined on $\mathfrak{B}$ in the following way. Let $\mu(B_0) = \mu(B_1) = 1/4$, $\mu(B_2) = 3/8$, $\mu(B_3) = 1/8$. Let $\nu(B_0) = \nu(B_3) = 3/8$ and $\nu(B_1) = \nu(B_2) = 1/8$. Clearly, $\mathrm{rng}(\mu) = \mathrm{rng}(\nu)$ and μ and ν are not isomorphic. Now, extend μ and ν to μ' and ν' defined on $\mathfrak{A}_c$ in such a way that $\mathrm{rng}(\mu') = \mathrm{rng}(\nu')$ and $\mu'(A) \notin \mathbb{Q}$ for every $A \in \mathfrak{A}_c \setminus \mathfrak{B}$. It is easy to see that it can be done and that it follows that μ' and ν' are not isomorphic.

However, there exists quite a nice coding of measures into countable subsets of $(0,1)$, at least for non–atomic strictly positive measures. Denote by naM the (Borel)

subspace of spM consisting of those elements of spM which induce non–atomic measures. Enumerate $2^{<\omega} \setminus \{\emptyset\} = \{s_n : n \in \omega\}$ in such a way that $|s_n| < |s_{n+1}|$ or $|s_n| = |s_{n+1}|$ and $s_n <_{lex} s_{n+1}$ for each n. For $s \in 2^{<\omega}$ define

$$\widehat{s} = \bigcup \{[t] : |t| = |s| \text{ and } t <_{lex} s\}.$$

Notice that $\widehat{\mathcal{T}} = \{\widehat{s_n} : n \in \omega\}$ generates $\mathfrak{A}_c$. Now, define $f : \mathsf{naM} \to [(0,1)]^\omega$ by

$$f(g)(n) = \mu_g(\widehat{s_n}),$$

where μ_g is the measure induced by $g \in \mathsf{naM}$. In other words, $r \in (0,1)$ is an element of $f(\mu)$ if there is a level l of $2^{<\omega}$ and a number $n < 2^l$ such that the sum of measures of first n elements of this level equals r (whatever is meant by "first n elements of level" as long as we have some fixed Borel procedure in mind).

Denote by $\equiv_r$ the equivalence relation of measures on the Cantor algebra such that $\mu \equiv_r \nu$ iff $\operatorname{rng}(f(\mu)) = \operatorname{rng}(f(\nu))$.

Theorem 3.3. *(1) f is onto a G_δ subset of $[(0,1)]^\omega$,*
(2) f is Borel,
(3) $\mu \equiv_r \nu$ implies $\mu \equiv \nu$ but the reverse implication does not hold,
(4) $\equiv_r$ is reducible to $=^+$.

Proof. To see (1) notice that a sequence is in the range of f if it is dense in $(0,1)$ (a G_δ condition) and satisfies a certain more complicated, though still G_δ, condition of "intertwining" ($0 < \boldsymbol{x_0} < 1$, $0 < \boldsymbol{x_1} < x_0 < \boldsymbol{x_2} < 1$, $0 < \boldsymbol{x_3} < x_1 < \boldsymbol{x_4} < x_0 < \boldsymbol{x_5} < x_2 < \boldsymbol{x_6} < 1$ and so on).

(2) is immediate.

To prove (3) assume that $\operatorname{rng}(f(\mu)) = \operatorname{rng}(f(\nu))$. We will define a metric isomorphism φ between $(\mathfrak{A}_c, \mu)$ and $(\mathfrak{A}_c, \nu)$. Since μ and ν are strictly positive, they are one–to–one on $\widehat{\mathcal{T}}$ and therefore the following definition of a measure–preserving bijection $\widehat{\varphi} : \widehat{\mathcal{T}} \to \widehat{\mathcal{T}}$ makes sense:

$$\widehat{\varphi}(\widehat{s}) = t \text{ iff } t \in \widehat{\mathcal{T}} \cap \nu^{-1}[\mu(\widehat{s})].$$

Using $\widehat{\varphi}$ one can define in a natural way a measure–preserving bijection $\varphi' : \mathcal{T} \to \mathcal{T}'$, where $\mathcal{T}'$ is a generating tree of $\mathfrak{A}_c$:

$$\varphi'(s_{n+1}) = \widehat{\varphi}(\widehat{s_{n+1}}) \setminus \widehat{\varphi}(\widehat{s_n})$$

if $|s_n| = |s_{n+1}|$ and

$$\varphi'(s_{n+1}) = \widehat{\varphi}(\widehat{s_{n+1}})$$

otherwise. Clearly φ' has a unique extension to an isomorphism $\varphi : \mathfrak{A}_c \to \mathfrak{A}_c$ such that $\nu(\varphi(a)) = \mu(a)$ for each $a \in \mathfrak{A}_c$.

To see the second part of (3) consider μ which is one–to–one on $\mathfrak{A}_c$ and ν defined by $\nu(a) = \mu(g(a))$, where g is a nontrivial automorphism of $\mathfrak{A}_c$. Then $\mu \equiv \nu$ but $\operatorname{rng}(f(\mu)) \neq \operatorname{rng}(f(\nu))$.

(1) and (2) imply that the equivalence relation $\equiv_r$ is reducible to $=^+$ restricted to a G_δ subset of $(0,1)^\omega$, which is itself reducible to $=^+$. □

3.4. **Problems.** We have shown that two equivalence relations similar to metric isomorphism of strictly positive measures on $\mathfrak{A}_c$ are Borel. However, since both of them seem to be much simpler than the metric isomorphism, we think that the metric isomorphism itself is probably more complicated, even if considered only on strictly positive non–atomic measures on $\mathfrak{A}_c$.

Problem 3.4. *What is the complexity of the equivalence relation of the metric isomorphism of strictly positive measures on $\mathfrak{A}_c$? What is the complexity of the metric isomorphism of strictly positive non–atomic measures on $\mathfrak{A}_c$?*

Theorem 3.2 suggests that the isomorphism of measures which are not necessarily strictly positive is most probably more complicated.

Problem 3.5. *What is the complexity of the equivalence relation of the metric isomorphism of all measures on $\mathfrak{A}_c$?*

Let us also indicate another possible area of research.

Superatomic Boolean algebras are known to have good classification (they are classified by (α, β), where α is the Cantor–Bendixon height and β is the number of atoms in the last Cantor–Bendixon derivative, cf. [4, Section 4]). Moreover, all measures on superatomic algebras are purely atomic, i.e. of the form

$$\mu = \sum_{n\in\omega} a_n \delta_{x_n}$$

for a sequence of real numbers $(a_n)_n$ and a sequence of points $(x_n)_n$ in the Stone space of the algebra. These facts motivate:

Problem 3.6. *Is there a (reasonable) classification of measures on countable superatomic Boolean algebras?*

4. A characterization of uniform regularity

In this section we will prove the following theorem.[3]

Theorem 4.1. *Assume that a Boolean algebra $\mathfrak{A}$ supports a uniformly regular non–atomic measure μ. Then $(\mathfrak{A}, \mu)$ is metrically isomorphic to a subalgebra of the Jordan algebra with the Lebesgue measure. Consequently, a Boolean algebra supports a non–atomic uniformly regular measure if and only if it is isomorphic to a subalgebra of the Jordan algebra $\mathcal{J}$ containing a dense Cantor subalgebra.*

At first, we show that the Boolean algebras supporting non–atomic uniformly regular measures can be seen as subalgebras of the Cohen algebra:

Proposition 4.2. *Assume that a Boolean algebra $\mathfrak{A}$ supports a non–atomic uniformly regular measure μ. Then $\mathfrak{A}$ is isomorphic to a subalgebra of the Cohen algebra.*

Before proving the above proposition we will show a more general fact. Recall that every finitely additive measure on a Boolean algebra $\mathfrak{A}$ has a unique extension to a measure on its σ–completion $\sigma(\mathfrak{A})$.

Proposition 4.3. *Assume that a Boolean algebra $\mathfrak{A}$ supports a measure μ and that $\mathfrak{B}$ is uniformly μ–dense in $\mathfrak{A}$. Then $(\mathfrak{A}, \mu)$ is metrically isomorphic to $(\mathfrak{A}', \mu')$, where $\mathfrak{A}' \subseteq \sigma(\mathfrak{B})$ and μ' is the unique extension of $\mu|_{\mathfrak{B}}$ to $\sigma(\mathfrak{B})$.*

Proof. Define a function $\varphi\colon \mathfrak{A} \to \mathfrak{C}$ in the following way:

$$\varphi(a) = \bigvee \{b \in \mathfrak{B} \colon b \le a\}.$$

We will prove that this function is a homomorphism. Everywhere below we assume that $b \in \mathfrak{B}$.

[3] We emphasize that homomorphisms throughout are not assumed to preserve infinitary operations.

- $\vee$. Observe that

  $$\bigvee\{b \in \mathfrak{B} \colon b \leq f \vee g\} \geq (\bigvee\{b \in \mathfrak{B} \colon b \leq f\} \vee \bigvee\{b \in \mathfrak{B} \colon b \leq g\}).$$

  Suppose that the above inequality is strict. Then, since $\mathfrak{B}$ is dense in $\sigma(\mathfrak{B})$, we would find nonzero $x \in \mathfrak{B}$ such that

  $$x \leq \varphi(f \vee g) \setminus (\varphi(f) \vee \varphi(g)).$$

  We have $x \wedge f \neq \mathbf{0}$ or $x \wedge g \neq \mathbf{0}$ (otherwise, we would have $x \wedge a = \mathbf{0}$ for every $a \leq f \vee g$, but this is impossible since $x \leq \varphi(f \vee g)$). Say $x \wedge f \neq \mathbf{0}$. As $\mathfrak{B}$ is dense in $\mathfrak{A}$, there is $b \in \mathfrak{B}$ such that $b \leq x \wedge f$. Then $b \leq f$ and $b \leq x$, a contradiction.
- ^{c}. Similarly, assume that

  $$\bigvee\{b' \in \mathfrak{B} \colon b' \leq f^c\} \neq \bigwedge\{b^c \colon b \leq f\}$$

  for some $f \in \mathfrak{A}$. That would mean that there are $b, b' \in \mathfrak{B}$ with $b \leq f$ and $b' \leq f^c$ such that either

  $$b' \nleq b^c,$$

  or

  $$b^c \nleq b'$$

  which is impossible.

Since $\mathfrak{B}$ is uniformly μ–dense in $\mathfrak{A}$, we have

$$\mu(a) = \mu'(\varphi(a))$$

for each $a \in \mathfrak{A}$. Clearly, φ is one–to–one, so φ is a metric monomorphism. $\square$

From the above proposition we can deduce Proposition 4.2. Indeed, if $\mathfrak{A}$ supports a uniformly regular non–atomic measure, then it has a dense countable family. We can assume that this family is a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. It is necessarily non–atomic and, therefore, isomorphic to the Cantor algebra $\mathfrak{A}_c$. We can thus assume that $\mathfrak{A}_c$ is a dense subalgebra of $\mathfrak{A}$. But the σ–completion of $\mathfrak{A}_c$ is the Cohen algebra.

We will strengthen Proposition 4.2. First, we introduce a general definition. Let μ be a measure defined on $\mathfrak{B}$. We say that the algebra

$$\mathcal{J}_\mu(\mathfrak{B}) = \{a \in \sigma(\mathfrak{B}) \colon \mu_*(a) = \mu^*(a)\}$$

is *a μ–Jordan extension of* $\mathfrak{B}$. Equivalently we can say that $\mathcal{J}_\mu(\mathfrak{B})$ consists of those Baire subsets of Stone($\mathfrak{B}$) whose boundaries are $\widehat{\mu}$–null (where $\widehat{\mu}$ is the unique extension of μ). The measure $\widehat{\mu}$ is strictly positive on $\mathcal{J}_\mu(\mathfrak{B})$ and $\mathfrak{B}$ is uniformly $\widehat{\mu}$–dense in $\mathcal{J}_\mu(\mathfrak{B})$. Notice also that $\mathfrak{B}$ is not uniformly $\widehat{\mu}$–dense in any subalgebra of $\sigma(\mathfrak{B})$ which is not included in $\mathcal{J}_\mu(\mathfrak{B})$.

Coming back to uniformly regular measures, the above remarks imply the following generalization of Proposition 4.2:

Proposition 4.4. *Every Boolean algebra supporting a uniformly regular measure is metrically isomorphic to a subalgebra of $\mathcal{J}_\mu(\mathfrak{A}_c)$ for some μ.*

With λ being the Lebesgue measure, the λ-Jordan extension of $\mathfrak{A}_c$ is in fact the well known Jordan algebra $\mathcal{J}$ coming from the Jordan measure on [0,1], see [14] and [24].

In Theorem 4.9 we will generalize Proposition 4.4 further by showing that $\mathcal{J}_\mu(\mathfrak{A}_c)$, which we will denote from now on by $\mathcal{J}_\mu$, is the same algebra for every non–atomic

measure μ on $\mathfrak{A}_c$ with respect to isomorphism. So, in fact every $\mathcal{J}_\mu$ is simply the Jordan algebra $\mathcal{J}$. Let us proceed towards the proof.

In our setting all elements of the Cohen algebra are countable unions of elements of the generating tree $\mathcal{T}$. The key tool in verifying if a given element of $\mathfrak{C}$ is in $\mathcal{J}_\mu$ are the partitions of $\mathbf{1}$ into elements of $\mathcal{T}$.

Definition 4.5. *We will say that a partition $\{a_n : n \in \omega\} \subseteq \mathcal{T}$ of **1** is* good for μ *(or, shortly, μ–good) if*

$$\sum_n \mu(a_n) = 1.$$

Proposition 4.6. *Consider a measure μ on the Cantor algebra $\mathfrak{A}_c$. If a partition $(a_n)_n$ is good for μ, then for every $M \subseteq \omega$*

$$\bigvee_{n\in M} a_n \in \mathcal{J}_\mu.$$

If $(a_n)_n$ is not good for μ, then for every infinite co–infinite $M \subseteq \omega$

$$\bigvee_{n\in M} a_n \notin \mathcal{J}_\mu.$$

Proof. Suppose first that $(a_n)_n$ is good for μ. Notice then that for any element $c \in \mathfrak{C}$ we have

$$\mu_*(c) = \sup\{\sum_{n\in\omega} \mu(b_n) \colon b_n \in \mathfrak{A}_c,\ (b_n)_n \text{ is pairwise disjoint, } \bigvee b_n = c\}.$$

To show that $\mu_*(c) = \mu^*(c)$ we have to prove that $\mu_*(c) + \mu_*(c^c) = 1$, but if $c = \bigvee_{n\in M} a_n$, then

$$1 \geq \mu_*(c) + \mu_*(c^c) \geq \sum_{n\in\omega} \mu(a_n) = 1,$$

since $(a_n)_n$ is good.

The second part of the proposition can be proved in the same way. $\square$

Notice that for every partition $\{a_n : n \in \omega\} \subseteq \mathcal{T}$ of $\mathbf{1}$ there are measures μ, ν, for which $(a_n)_n$ is μ–good and not ν–good. Indeed, define the measures on (a_n) by $\mu(a_n) = 1/2^{n+1}$ and $\nu(a_n) = 1/2^{n+2}$ for every n and then extend them to strictly positive and non–atomic measures on $\mathfrak{A}_c$. It is also easy to see that for every measure μ on $\mathfrak{A}_c$ and $\varepsilon > 0$ there is a partition $(a_n)_n$ of $\mathbf{1}$ such that $\Sigma_n \mu(a_n) < \varepsilon$. In particular, there is no measure such that $\mathcal{J}_\mu = \mathfrak{C}$ and therefore there is no strictly positive uniformly regular measure on the Cohen algebra. Of course, the Cohen algebra is Kelley, so we have the following

Corollary 4.7. *There are Boolean algebras with a countable dense set (and, thus, Kelley) which do not support a uniformly regular measure.*

Now we will show that a Boolean algebra $\mathcal{J}_\mu$ is isomorphic to $\mathcal{J}_\lambda$ for every non–atomic measure μ. In analogy with what we have done with partitions of unity, for a set $x \in \mathcal{J}_\mu$ we say that $\{a_n : n \in \omega\} \subseteq \mathcal{T}$ is a *μ–good partition of x* if $\sum_n \mu(a_n) = \widehat{\mu}(x)$, where $\widehat{\mu}$ is the unique extension of μ to $\mathcal{J}_\mu$.

Lemma 4.8. *Let μ be a non–atomic measure on $\mathfrak{A}_c$ and $\widehat{\mu}$ its (unique) extension to $\mathcal{J}_\mu$. For every $x \in \mathcal{J}_\mu$ and every $0 < \varepsilon < \mu(x)$ there is $j \in \mathcal{J}_\mu$ such that $j \leq x$ and $\widehat{\mu}(j) = \varepsilon$.*

Proof. Assume without loss of generality that $x = \mathbf{1}$. Since μ is non-atomic on $\mathfrak{A}_c$, it is easy to see that there is a μ–good partition $(a_n)_n$ of $\mathbf{1}$ such that

$$\sum_{n \in N} \mu(a_n) = \varepsilon,$$

where N is the set of odd numbers. Then $j = \bigvee_{n \in N} a_n \in \mathcal{J}_\mu$ and, of course, $\widehat{\mu}(j) = \varepsilon$. $\square$

We will use this fact to prove the following theorem.

Theorem 4.9. *For every non–atomic measure μ on the Cantor algebra, the μ-Jordan extension algebra $(\mathcal{J}_\mu, \mu)$ is (metrically) isomorphic to $(\mathcal{J}, \lambda)$.*

Proof. Let μ be a measure on $\mathfrak{A}_c$, and let $\widehat{\mu}$ be its extension to $\mathcal{J}_\mu$. We identify the $\mathfrak{A}_c$'s generating tree $\mathcal{T}$ with $2^{<\omega}$ here. We will find a subalgebra $\mathfrak{A}'$ of $\mathcal{J}_\mu$ such that

- there is an isomorphism $\varphi\colon \mathfrak{A}_c \to \mathfrak{A}'$;
- $\mathfrak{A}'$ is dense in $\mathcal{J}_\mu$;
- for every $s \in 2^n$ we have $\widehat{\mu}(\varphi(s)) = 1/2^n$.

Fix an enumeration $\mathfrak{A}_c \setminus \{\emptyset\} = \{d_n \colon n \in \omega\}$. We will inductively define an isomorphism φ between $\mathcal{T}$ and a subset of $\mathcal{J}_\mu$. Let $\varphi(\emptyset) = \mathbf{1}$ and let $m_0 = 0$. Assume that we have defined $\varphi(s)$ for every $s \in 2^i$, $i \leq m_n$ in such a way that $\{\varphi(s) \colon s \in 2^{m_n}\}$ is a partition of $\mathbf{1}$ which is dense under $\{d_1, \dots, d_n\}$ and $\widehat{\mu}(\varphi(s)) = 1/2^i$ for $s \in 2^i$.

For some $s \in 2^{m_n}$ we have $\varphi(s) \wedge d_{n+1} \neq \mathbf{0}$. Using Lemma 4.8 we can find an element y of $\mathcal{J}_\mu$ such that

- $\widehat{\mu}(y) = 1/2^l$ for some $l > m_n$;
- $y \leq \varphi(s) \wedge d_{n+1}$.

Now define φ on 2^l in such a way that

- $\varphi(t) = y$ for some $t \in 2^l$ extending s;
- $\widehat{\mu}(\varphi(s)) = 1/2^l$ for $s \in 2^l$;
- $\{\varphi(s) \colon s \in 2^l\}$ is a partition of $\mathbf{1}$ refining $\{\varphi(s) \colon s \in 2^{m_n}\}$.

Put $m_{n+1} = l$. Define $\mathfrak{A}'$ to be the algebra generated by $\{\varphi(s) \colon s \in 2^{<\omega}\}$ and notice that $\mathfrak{A}'$ is dense in $\mathfrak{A}$ and $\widehat{\mu}(\varphi(s)) = 1/2^{|s|}$ for $s \in 2^{<\omega}$. Now, consider the Jordan extension $\mathcal{J}'_{\widehat{\mu}}$ of $\mathfrak{A}'$.

CLAIM: $\mathcal{J}_\mu = \mathcal{J}'_{\widehat{\mu}}$.

It is enough to show that every μ-good partition of $\mathbf{1}$ into elements of the generating tree $\mathcal{T}$ can be refined to a μ'–good partition of $\mathbf{1}$ into elements of $\{\varphi(s) \colon s \in 2^{<\omega}\}$ and vice versa.

Let $\{a_n \colon n \in \omega\} \subseteq 2^{<\omega}$ be a μ–good partition of $\mathbf{1}$. Then, using the fact that $\mathfrak{A}'$ is dense under $\mathfrak{A}$, for every n we can find a μ'–good partition $\{b^n_m \colon m \in \omega\} \subseteq \{\varphi(s) \colon s \in 2^{<\omega}\}$ of a_n. Thus, $\{b^n_m \colon m, n \in \omega\} \subseteq \{\varphi(s) \colon s \in 2^{<\omega}\}$ is a μ'–good partition of $\mathbf{1}$.

The reverse implication can proved in the same way. $\square_{Claim}$

Now, we want to show that there is a metric isomorphism $\psi\colon \mathcal{J}_\mu \to \mathcal{J}$. For $s \in \mathcal{T}$ define

$$\psi(\varphi(s)) = s.$$

Of course $\psi = \varphi^{-1}$ so it is an isomorphism. Additionally $\widehat{\mu}(s) = 1/2^{|s|} = \lambda(s)$, so ψ is measure preserving. We can extend ψ to

$$\psi\colon \mathcal{J}'_{\widehat{\mu}} \to \mathcal{J}.$$

But $\mathcal{J}'_{\widehat{\mu}} = \mathcal{J}_\mu$, so we are done. □

The Maharam theorem implies that every complete Boolean algebra supporting a non–atomic σ–additive separable measure is metrically isomorphic to the Random algebra with the Lebesgue measure. Theorem 4.9 gives us an analogous theorem for uniformly regular measures. There is no *complete* Boolean algebra supporting a uniformly regular measure, but we can see a property of being isomorphic to the Jordan algebra as a property of *being as close to completeness as possible without loosing uniform regularity*.

Theorem 4.9 allows us to complete the proof of Theorem 4.1.

Proof. (of Theorem 4.1) Assume $\mathfrak{A}$ supports a non–atomic uniformly regular measure μ. By Proposition 4.2 $(\mathfrak{A}, \mu)$ is metrically isomorphic to a subalgebra of $\mathcal{J}_\mu$ with $\widehat{\mu}$, the unique extension of μ to $\sigma(\mathfrak{A})$. By Theorem 4.9 $\mathcal{J}_\mu$ is metrically isomorphic to $(\mathcal{J}, \lambda)$, so $(\mathfrak{A}, \mu)$ is metrically isomorphic to a subalgebra of $\mathcal{J}$ with λ.

Assume now that a Boolean algebra $\mathfrak{A}$ is a subalgebra of $\mathcal{J}$ and that it has a dense Cantor subalgebra $\mathfrak{B}$. Then $\mathfrak{B}$ is uniformly λ–dense in $\mathfrak{A}$, so $\mathfrak{A}$ supports a uniformly regular measure. □

Because of the complexity of non σ–complete Boolean algebras, it seems that every characterization here inevitably has to involve subalgebras, like in Theorem 4.1. So, we can only hope to fully characterize Boolean algebras supporting measures which are *maximal* with respect to some property.

The fact analogous to the first part of Theorem 4.1 for separable measures is an easy application of Maharam theorem.

Proposition 4.10. *Assume a Boolean algebra $\mathfrak{A}$ supports a non–atomic separable measure μ. Then $(\mathfrak{A}, \mu)$ is metrically isomorphic to a subalgebra of the Random algebra with the Lebesgue measure.*

Proof. Consider the (unique) extension $\widehat{\mu}$ of μ to $\mathfrak{B} = \mathrm{Borel}(\mathrm{Stone}(\mathfrak{A}))$. It is non–atomic and separable, so the measure $\widehat{\mu}$ defined on $\mathfrak{B}/\{b\colon \widehat{\mu}(b) = 0\}$ is, by the Maharam theorem, metrically isomorphic to the Lebesgue measure on the Random algebra. Since μ is strictly positive, the function $\varphi\colon \mathfrak{A} \to \mathfrak{B}/\{b\colon \widehat{\mu}(b) = 0\}$ defined by

$$\varphi(a) = [a]_{\widehat{\mu}}$$

is a monomorphism. □

However, we cannot conclude from the above that every subalgebra of the Random algebra supports a non–atomic separable measure ($\mathrm{Free}(\omega_1)$ is one of the counterexamples). It is also unclear what assumption (analogous to containing a dense Cantor subalgebra in Theorem 4.1) should be added to Proposition 4.10 to obtain a characterization of algebras supporting non–atomic separable measures. Thus, uniform regularity seems to be a more convenient notion for our purposes.

5. Higher cardinal versions of uniform regularity

The positive classification results that we obtained when replacing separability by uniform regularity motivate us to consider higher cardinal versions of uniform regularity. The results that we have on this subject are preliminary and the further research is planned in the future. Nevertheless, the partial results we do have seem worth mentioning in this brief section.

For a measure μ on a Boolean algebra $\mathfrak{B}$ we can define the cardinal number

$$\mathrm{ur}(\mu) = \min\{\kappa : \text{there is a uniformly } \mu - \text{dense family } \mathcal{F} \subseteq \mathfrak{B} \text{ with } |\mathcal{F}| = \kappa\}.$$

This is a well defined cardinal invariant since $\mathfrak{B}$ is uniformly μ-dense in $\mathfrak{B}$. Moreover, if the algebra is atomless, this cardinal invariant is always an infinite cardinal. In relation with known cardinal invariants of Boolean algebras and measures, notice that clearly $\mathrm{ur}(\mu)$ is $\geq$ than the Maharam type of μ and also than the pseudoweight $\pi(\mathfrak{B})$, which is defined as the smallest cardinality of a set A of positive elements in $\mathfrak{B}$ such that for all $b \in \mathfrak{B} \setminus \{\mathbf{0}\}$ there is $a \in A$ such $a \leq b$.

Lemma 5.1. *Suppose that μ is a non-atomic strictly positive measure on a Boolean algebra $\mathfrak{B}$. Then there is a partition of unity $\{a_n \colon n < \omega\}$ such that for every n and every $\mathbf{0} \neq b \leq a_n$ we have that $ur(\mu \,|\, b) = ur(\mu \,|\, a_n)$.*

Proof. Let us call an element $a \in B$ *uniform* if for every $\mathbf{0} \neq b \leq a$ we have that $\mathrm{ur}(\mu \,|\, b) = \mathrm{ur}(\mu \,|\, a)$. We claim that the set of uniform elements is dense in $\mathfrak{B}$. If not, we can find a sequence $\{b_n \colon n < \omega\}$ of elements of $\mathfrak{B}$ such that $\mathrm{ur}(\mu \,|\, b_n) > \mathrm{ur}(\mu \,|\, b_{n+1})$, which gives an infinite decreasing sequence of cardinals. Now it suffices to take a maximal antichain in the family of uniform elements. Such an antichain must be countable, by the ccc property of the algebra, and it must be a partition of unity by the density of the set of uniform elements. $\square$

The above Lemma parallels the reduction in Maharam's theorem to homogeneous measures. We shall say that a measure μ on an algebra $\mathfrak{B}$ is *uniformly κ–regular* if $\mathbf{1}$ is a uniform element and $\mathrm{ur}(\mu) = \kappa$. We would like to classify pairs $(\mathfrak{B}, \mu)$ where μ is a uniformly κ-regular measure on $\mathfrak{B}$, for various κ. For $\kappa = \aleph_0$ we have already determined that these pairs are exactly subalgebras of the Jordan algebra with the Lebesgue measure.

We need to introduce higher analogues of the Jordan algebra.

Let us denote by λ_κ the usual product measure on 2^κ and by $\mathfrak{A}_\kappa$ the clopen algebra of 2^κ. We shall also denote by $\mathcal{T}_\kappa$ the family of basic clopen sets in 2^κ, which can be identified with the free algebra on κ generators. Then Kakutani's theorem says that λ_κ is obtained as an extension of the measure on $\mathcal{T}_\kappa$ which to each basic clopen set of the form $[s]$ assigns $1/2^{|\mathrm{dom}(s)|}$. The point is that all Borel sets are indeed measurable when we extend the measure, even though the σ-completion of $\mathfrak{A}_\kappa$ only gives us the Baire algebra $\mathfrak{B}_\kappa$, which in general does not contain all Borel subsets of 2^κ. We obtain the same algebra if we work with $[0,1]^\kappa$ in place of 2^κ, and we denote both of these measures as λ_κ.

Now we shall define analogues of the Jordan algebra. For any κ let $\mathcal{J}^\kappa$ be the algebra of those open sets in $[0,1]^\kappa$ whose boundaries have the λ_κ measure 0, so $\mathcal{J}^\omega = \mathcal{J}$. In other words, $\mathcal{J}^\kappa = \mathcal{J}_{\lambda_\kappa}(\mathrm{Free}(\kappa))$. We might hope to prove that a Boolean algebra supports a uniformly κ–regular measure if and only if it is isomorphic to a subalgebra of the algebra $\mathcal{J}^\kappa$ containing a dense copy of some fixed nice algebra $\mathfrak{A}^*_\kappa$. Note however that certainly this hypothetic $\mathfrak{A}^*_\kappa$ cannot be

$\mathfrak{A}_\kappa$, since as soon as a Boolean algebra contains a copy of $\mathcal{T}_\kappa$, it has an independent sequence of length κ and hence it supports a measure of type κ. However, for $\kappa > \omega$ there may be Boolean algebras $\mathfrak{B}$ that do not support a measure of type κ but all measures μ on $\mathfrak{B}$ satisfy $\mathrm{ur}(\mu) \geq \kappa$, see Section 6 for examples. It is in fact more reasonable to widen our acceptance criterion from a fixed $\mathfrak{A}^*_\kappa$ to a quotient of $\mathfrak{A}_\kappa$, as suggested by the following observation which generalizes Proposition 4.2.

Proposition 5.2. *If a Boolean algebra supports a uniformly κ-regular (non-atomic) measure, then it is isomorphic to a subalgebra of the σ-completion of a quotient of* $\mathrm{Free}(\kappa)$.

Proof. Let $\mathfrak{A}$ be a Boolean algebra that supports a uniformly κ-regular measure and let $\mathfrak{B}$ be a uniformly μ-dense subset of $\mathfrak{A}$ size κ. We may assume that $\mathfrak{B}$ is a Boolean algebra. Therefore $\mathfrak{B}$ is isomorphic to a quotient $\mathfrak{B}'$ of $\mathrm{Free}(\kappa)$. Let $\mathfrak{C}'$ be the σ-completion of $\mathfrak{B}'$, we shall show that $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{C}'$. To do this, we use exactly the same definition of f as in the proof of Proposition 4.2 and the proof that f is a homomorphism remains the same. □

If we wish to generalise further, we need to develop the analogues of the uniqueness of the Jordan algebra. The proof we had in the separable case rested upon the uniqueness of the Cantor algebra. In the higher-dimensional case we cannot hope for that, but perhaps we can obtain uniqueness restricted to algebras that have the same dense set. This research brings us out of the scope of the present article and we plan it for the future work.

6. Separability versus uniform regularity

In Section 4 we presented a characterization of Kelley algebras supporting uniformly regular measures. Finding a characterization of Kelley algebras supporting separable measures seems to be a more difficult task (see e.g. [5]) as well as finding a characterization of Kelley algebras carrying only separable measures. The latter is at least possible under $\mathsf{MA}(\omega_1)$:

Theorem 6.1. ([7, Theorem 9]) *If a Boolean algebra carries only separable measures, then it is small. Under $\mathsf{MA}(\omega_1)$ the converse implication holds.*

However, consistently there are small Boolean algebras carrying a non–separable measure (see e.g. the literature listed in [28] after Theorem 6.4).

The natural question is if one can use the result from the previous section to get some information about properties of Kelley algebras supporting only separable measures. Some connections of uniform regularity and separability are obvious: e.g. uniformly regular measures are separable. The following fact indicates that there are some more subtle relationships at work.

Theorem 6.2. ([2, Theorem 4.6]) *Every Boolean algebra carries either a non–separable measure or a measure which is uniformly regular.*

Recently, Mikołaj Krupski proved the above theorem in a more general setting, see [19].

One should point out here that (consistently) there are small Boolean algebras without uniformly regular measures. Indeed, Talagrand ([26]) used CH to construct a small Gronthendieck space K, i.e. a space such that there are no non–trivial (i.e. not weakly convergent) weakly* convergent sequences of measures on

K. Grothendieck property implies that $P(K)$ does not have G_δ points, and thus (by [25, Proposition 2]) K does not carry a uniformly regular measure. Talagrand's example is zero–dimensional and cannot be continuously mapped onto $[0,1]^{\omega_1}$. Thus, it is the Stone space of a small Boolean algebra without a uniformly regular measure. Such an example cannot be, however, constructed without additional axioms, because of Theorem 6.1 and Theorem 6.2.

Notice also, that the alternative in Theorem 6.2 is, by no means, exclusive. There are many Boolean algebras with both non–separable and uniformly regular measures (the Jordan algebra can serve as an example here).

For our purposes a *strictly positive* version of Theorem 6.2 would be most desirable. However, it turned out that we cannot hope for that:

Theorem 6.3. *There is a Kelley algebra supporting only separable measures but no uniformly regular one.*

We will prove this theorem building on ideas contained in [1]. Bell constructed in this paper a Boolean space which is separable, which does not have a countable π–base and whose algebra of clopen subsets is small.

The space presented below is similar to the space constructed by Bell. However, our approach is different and, at least for our purposes, simpler than that of Bell. In particular, it allows us to prove that each measure supported by this space is separable.

First, we introduce some notation. For $A \subseteq \omega$ let $A^0 \subseteq 2^\omega$ be the set of the form

$$A^0 = \{x \in 2^\omega : \forall n \in A \ \ x(n) = 0\}.$$

For a family $\mathcal{A} \subseteq P(\omega)$ let

$$\mathcal{A}^0 = \{A^0 : A \in \mathcal{A}\}.$$

If $\mathcal{A} \subseteq P(\omega)$, then let $\mathfrak{A}(\mathcal{A}) \subseteq P(2^\omega)$ be the Boolean algebra generated by $\mathcal{A}^0$ and let $K(\mathcal{A})$ be the Stone space of this algebra. For $A \subseteq \omega$ let

$$A^1 = \{x \in 2^\omega : \exists n \in A \ \ x(n) = 1\}.$$

Of course $A^1 = (A^0)^c$ for every $A \subseteq \omega$.

Let us collect some immediate observations:

Proposition 6.4. 1) *If* Fin *is the set of finite subsets of* ω*, then* Fin^0 *generates* $\mathrm{Clop}(2^\omega)$*;*

2) *If* $\mathrm{Fin} \subseteq \mathcal{A}$*, then* $\mathfrak{A}(\mathcal{A})$ *is an extension of the Cantor algebra (and there is a continuous function from* $K(\mathcal{A})$ *onto* 2^ω*). Every* $x \in 2^\omega$ *can be interpreted as a closed subset of* $K(\mathcal{A})$*. Namely, for* $x \in 2^\omega$ *let* F_x *be the set of all ultrafilters on* $\mathfrak{A}(\mathcal{A})$ *extending the filter generated by*

$$\{\{n\}^0 : x(n) = 0\} \cup \{\{n\}^1 : x(n) = 1\};$$

For $\mathcal{A} \subseteq P(\omega)$ the family of elements of the form

$$A_0^0 \cap A_1^0 \cap \cdots \cap A_k^0 \cap A_{k+1}^1 \cap \cdots \cap A_n^1$$

is a base of $K(\mathcal{A})$. Every element of $\mathfrak{A}(\mathcal{A})$ is a finite union of sets of this form. Since $A^0 \cap B^0 = (A \cup B)^0$, if $\mathcal{A}$ is closed under taking finite unions, then elements of the above base can be written in a simpler form:

$$A^0 \cap A_0^1 \cap \ldots \cap A_n^1$$

for $A, A_0, \ldots, A_n \in \mathcal{A}$.

Before pointing out which particular family $\mathcal{A}$ we will consider, we prove two general theorems concerning spaces $K(\mathcal{A})$.

Proposition 6.5. *Let* $\mathrm{Fin} \subseteq \mathcal{A} \subseteq P(\omega)$. *Then there is a countable family of closed subsets* $\mathcal{F}$ *of* $K(\mathcal{A})$ *such that for every nonempty open set* U *in* $K(\mathcal{A})$ *there is* $F \in \mathcal{F}$ *such that* $F \subseteq U$. *Consequently,* $K(\mathcal{A})$ *is separable.*

Proof. Let
$$\mathcal{F} = \{F_x \colon x \in 2^\omega,\ x(n) = 1 \text{ for finitely many } n\text{'s}\},$$
where F_x is as in Proposition 6.4 (2). Let $U \subseteq K(\mathcal{A})$ be an open subset. Without loss of generality we can assume that it is of the form
$$U = A_0^0 \cap A_1^0 \cap \ldots \cap A_k^0 \cap A_{k+1}^1 \cap \ldots \cap A_n^1,$$
for $A_1, \ldots, A_n \in \mathcal{A}$. If U is nonempty, then for every $i > k$ we have
$$B_i = A_i \setminus (A_0 \cup \cdots \cup A_k) \neq \emptyset.$$
Pick $n_i \in B_i$ for every $k < i \leq n$. Let $x \in 2^\omega$ be such that $x(n_i) = 1$ for every i and $x(n) = 0$ if there is no i such that $n = n_i$. Then $F_x \in \mathcal{F}$ and $F_x \subseteq U$. $\square$

Proposition 6.6. *Let* $\mathrm{Fin} \subseteq \mathcal{A} \subseteq P(\omega)$ *and assume that* $\mathcal{A}$ *is closed under finite unions. Suppose that* $\mathcal{A}$ *does not have a cofinite family of cardinality* λ, *i.e. for every* $\mathcal{A}_0 \subseteq \mathcal{A}$ *of size* λ *there is* $B \in \mathcal{A}$ *such that* $B \setminus A \neq \emptyset$ *for every* $A \in \mathcal{A}_0$. *Then* $K(\mathcal{A})$ *does not have a* π*–base of size* λ.

Proof. Suppose $\mathcal{V}$ is a π–base of $K(\mathcal{A})$. We can assume that it consists of sets of the form
$$V = A_V^0 \cap A_0^1 \cap \ldots \cap A_n^1$$
for $A_V, A_0, \ldots, A_n \in \mathcal{A}$. Assume that $B \in \mathcal{A}$ is infinite and $V \in \mathcal{V}$ is such that $V \subseteq B^0$. Clearly, $B \subseteq A_V$. So, if we let $\mathcal{A}_0 = \{A_V \colon V \in \mathcal{V}\}$, then

- $|\mathcal{A}_0| \leq |\mathcal{V}|$,
- for every $B \in \mathcal{A}$ there is $A \in \mathcal{A}_0$ such that $B \subseteq A$, so $\lambda < |\mathcal{A}_0|$.

Therefore, $\lambda < |\mathcal{V}|$. $\square$

Notice that if a family $\mathcal{A}$ contains an uncountable pairwise almost disjoint family $(A_\alpha)_{\alpha<\omega_1}$, then $(A_\alpha^0)_{\alpha<\omega_1}$ forms an uncountable independent sequence in $\mathfrak{A}(\mathcal{A})$ and consequently $\mathfrak{A}(\mathcal{A})$ is big. So, if we want to construct a Kelley algebra which is small (to omit supporting a non–separable measure, cf. Theorem 6.1), we have to use a family which does not contain many pairwise almost disjoint sets.

The natural example of such a family satisfying also the conditions of Theorem 6.6 is the following. Let $\{T_\alpha \colon \alpha < \omega_1\} \subseteq P(\omega)$ be such that $T_0 = \emptyset$ and for every $\alpha < \beta < \omega_1$ the set $T_\alpha \setminus T_\beta$ is finite and $T_\beta \setminus T_\alpha$ is infinite. Shortly speaking, $(T_\alpha)_\alpha$ is a strictly $\subseteq^*$–increasing tower. Let $\mathcal{T}$ consists of those sets $T \subseteq \omega$ such that $T = T_\alpha \cup F$ for some $\alpha < \omega_1$ and some finite $F \subseteq \omega$.

Notice that $\mathcal{T}$ satisfies the assumptions of Proposition 6.6, so $K(\mathcal{T})$ is a separable space without a countable π–base. Now we will prove that all measures on $K(\mathcal{T})$ are separable.

Recall that if μ is a non–separable measure on a Boolean algebra $\mathfrak{A}$, then we can find an uncountable family $\mathcal{A}$ of generators of $\mathfrak{A}$ and $\varepsilon > 0$ such that
$$\mu(A \triangle B) > \varepsilon$$

for every distinct A, $B \in \mathcal{A}$. Otherwise, we could find a countable family $\mathcal{B}$ which is μ–dense in the set of generators of $\mathfrak{A}$. But then the (countable) Boolean algebra generated by $\mathcal{B}$ would be μ–dense in $\mathfrak{A}$, and so μ would be separable.

Theorem 6.7. *Every measure on $\mathfrak{A}(\mathcal{T})$ is separable.*

Proof. Suppose toward a contradiction that there is a non–separable measure μ on $\mathfrak{A}(\mathcal{T})$. Then using the above remark assume that

$$\mu(T^1_\alpha \bigtriangleup T^1_\beta) > \varepsilon$$

for every $\alpha < \beta < \omega_1$. We can do it since $\{T^1 \colon T \in \mathcal{T}\}$ generates $\mathfrak{A}(\mathcal{T})$ and we can consider a subalgebra of $\mathfrak{A}(\mathcal{T})$ if necessary.

For $\alpha < \omega_1$ denote

$$\rho(T_\alpha) = \sup\{\mu(F^1) \colon F \in \mathrm{Fin}, F \subseteq T_\alpha\}.$$

Notice that $\rho(T_\alpha) = \nu_*(T^1_\alpha)$, where $\nu = \mu|\mathfrak{A}(\mathrm{Fin}^1)$, and that $\rho(T_\alpha) \leq \mu(T^1_\alpha)$ for every α.

Considering a sub–tower of $(T_\alpha)_{\alpha<\omega_1}$ of height ω_1 if necessary, we can assume that $|\mu(T^1_\alpha) - \mu(T^1_\beta)| < \varepsilon/3$ and $\rho(T_\alpha) > \sup_\alpha \rho(T_\alpha) - \varepsilon/6$ for every α, $\beta < \omega_1$.

CLAIM 1. We can assume that the tower $(T_\alpha)_{\alpha<\omega_1}$ has the following property (*): for every finite $F \subseteq \omega$ either there is no $\alpha < \omega_1$ such that $F \subseteq T_\alpha$ or $F \subseteq T_\alpha$ for uncountably many α's.

Enumerate $\{F_n \colon n \in \omega\}$ the set of those finite sets which are included in T_α for at most countably many α's. Let $\alpha_n = \sup\{\alpha \colon F_n \subseteq T_\alpha\}$ and let $\gamma = \sup_n \alpha_n$. It is clear that $(T_\alpha)_{\alpha>\gamma}$ is a tower of height ω_1 with the property (*). So we can assume without loss of generality that $\gamma = 0$ and $(T_\alpha)_{\alpha<\omega_1}$ has property (*).

CLAIM 2. For every $\alpha < \beta$ we have

$$\mu((T_\alpha \cup T_\beta)^1) > \mu(T^1_\beta) + \varepsilon/3.$$

Indeed,

$$\varepsilon < \mu(T^0_\alpha \bigtriangleup T^0_\beta) = \mu(T^0_\alpha) + \mu(T^0_\beta) - 2\mu((T_\alpha \cup T_\beta)^0).$$

Hence

$$2\mu((T_\alpha \cup T_\beta)^0) < \mu(T^0_\alpha) + \mu(T^0_\beta) - \varepsilon \leq 2\mu(T^0_\beta) + \varepsilon/3 - \varepsilon$$

and

$$\mu((T_\alpha \cup T_\beta)^0) < \mu(T^0_\beta) - \varepsilon/3.$$

Since $(T_\alpha \cup T_\beta)^1 = ((T_\alpha \cup T_\beta)^0)^c$ we have

$$\mu((T_\alpha \cup T_\beta)^1) > \mu(T^1_\beta) + \varepsilon/3.$$

CLAIM 3. If $\alpha < \beta$ and $F = T_\alpha \setminus T_\beta$, then for every nonempty finite $G \subseteq T_\beta$ we have $\mu((F \cup G)^1) > \mu(G^1) + \varepsilon/3$.

Using Claim 2 we have

$$\mu(T^1_\beta) + \mu(F^1) - \mu(T^1_\beta \cap F^1) = \mu(T^1_\alpha \cup T^1_\beta) = \mu((T_\alpha \cup T_\beta)^1) > \mu(T^1_\beta) + \varepsilon/3.$$

Thus

$$\mu(F^1) - \mu(T^1_\beta \cap F^1) > \varepsilon/3$$

and for every finite $G \subseteq T_\beta$

$$\mu(F^1) > \mu(G^1 \cap F^1) + \varepsilon/3, \quad \text{so} \quad \mu(F^1 \setminus G^1) > \varepsilon/3.$$

Finally

$$\mu((F \cup G)^1) = \mu(F^1 \cup G^1) > \mu(G^1) + \varepsilon/3,$$

and Claim 3 is proved.

Now, let $G \subseteq T_0$ be a finite set such that $\mu(G^1) > \rho(T_0) - \varepsilon/6$. Because of property (*) there is $\beta < \omega_1$ such that $G \subseteq T_\beta$. Let $F = T_0 \setminus T_\beta$. Then, using Claim 3 we have

$$\mu((F \cup G)^1) > \mu(G^1) + \varepsilon/3 > \rho(T_0) - \varepsilon/6 + \varepsilon > \sup_\alpha \rho(T_\alpha) - 2\varepsilon/6 + \varepsilon/3 = \sup_\alpha \rho(T_\alpha).$$

But $F \cup G \subseteq T_0$, so $\mu((F \cup G)^1) \leq \rho(T_0)$, a contradiction. □

Now we are finally ready to present a proof of Theorem 6.3:

Proof. Since $K(\mathcal{T})$ is separable, $\mathfrak{A}(\mathcal{T})$ supports a strictly positive measure. Since $K(\mathcal{T})$ does not have a countable π–base, it cannot support a uniformly regular measure. Finally, Theorem 6.7 implies that all measures on $\mathfrak{A}(\mathcal{T})$ are separable. □

We finish this section with two remarks, not connected directly to measures. First, we show that the fact that $\mathfrak{A}(\mathcal{T})$ does not have an uncountable independent sequence can be deduced in a slightly simpler way.

Theorem 6.8. *The Boolean algebra* $\mathfrak{A}(\mathcal{T})$ *is small.*

Proof. We will prove that every filter on $\mathfrak{A}(\mathcal{T})$ can be extended to an ultrafilter by countably many sets. This would imply that every closed subset of $K(\mathcal{T})$ contains a point with a relatively countable π–character and, because of Shapirovsky's theorem (see [28, Theorem 6.1]) that $\mathfrak{A}(\mathcal{T})$ is small.

First, observe that if $\mathcal{A} \subseteq P(\omega)$, $\mathcal{F}$ is a filter on $\mathfrak{A}(\mathcal{A})$ and there is no $A \in \mathcal{A}$ such that $\mathcal{F}$ can be extended by A^0, then $\mathcal{F}$ is an ultrafilter. Of course, the same holds true for the sets of the form A^1.

Let $\mathcal{F}$ be a filter on $\mathfrak{A}(\mathcal{T})$. Notice that without loss of generality we can assume that $\{n\}^0 \in \mathcal{F}$ or $\{n\}^1 \in \mathcal{F}$ for every $n \in \omega$, extending $\mathcal{F}$ by at most countably many sets, if necessary.

For $T \in \mathcal{T}$ say that the level of T is α if $T =^* T_\alpha$ (denote it by $\mathrm{lv}(T) = \alpha$). Assume $\mathcal{F}$ is not an ultrafilter and let γ be the minimal number such that there is S of level γ such that $\mathcal{F}$ can be extended by S^1. Notice that since $S^1 \notin \mathcal{F}$ the set $\{n\}^0 \in \mathcal{F}$ for every $n \in S$.

Extend $\mathcal{F}$ to $\mathcal{F}'$ by S^1. Then extend $\mathcal{F}'$ to $\mathcal{F}''$ by countably many sets in such a way that $\mathcal{F}''$ cannot be extended by any element of the set

$$\{T^0 \colon \mathrm{lv}(T) \leq \gamma\}.$$

It can be done since the above set is countable.

We will show that $\mathcal{F}''$ is an ultrafilter by showing that it cannot be extended by a set T^0 for $T \in \mathcal{T}$. Indeed, let $T \in \mathcal{T}$. If $\mathrm{lv}(T) \leq \gamma$, then either $T \in \mathcal{F}''$ or $\mathcal{F}''$ cannot be extended by T. If $\mathrm{lv}(T) > \gamma$, then the set $S \setminus T$ is finite. Moreover $(S \setminus T)^0 \in \mathcal{F}$. So, the set $T^0 \cap (S \setminus T)^0 \supseteq S^0$ cannot be added to $\mathcal{F}''$, and thus $\mathcal{F}''$ cannot be extended by T^0. □

The following theorem is also worth mentioning in this context. A Boolean algebra is minimally generated if there is a maximal chain in the lattice of its subalgebras and this chain is well–ordered (for a more intuitive but longer definition, see e.g. [18, Section 0]). The class of minimally generated Boolean algebras is

quite a big sub–class of small Boolean algebras. In [18] Koppelberg posed even the question if those classes are equal (she answered it negatively in [17, Example 1]). Minimally generated Boolean algebras are interesting from the point of view of measure theory because they carry only separable measures. Moreover, the following theorem holds.

Theorem 6.9. ([2, Theorem 4.15]) *Every minimally generated Kelley algebra supports a uniformly regular measure.*

It follows that the Boolean algebra constructed above is an another example of a small Boolean algebra which is not minimally generated.

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